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Geometric approach to nonlinear coherent states using the Higgs model for harmonic oscillator

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Abstract

In this paper, we investigate the relation between the curvature of the physical space and the deformation function of the deformed oscillator algebra using the nonlinear coherent states approach. For this purpose, we study two-dimensional harmonic oscillators on the flat surface and on a sphere by applying the Higgs model. With the use of their algebras, we show that the two-dimensional oscillator algebra on a surface can be considered as a deformed one-dimensional oscillator algebra where the effect of the curvature of the surface appears as a deformation function. We also show that the curvature of the physical space plays the role of deformation parameter. Then we construct the associated coherent states on the flat surface and on a sphere and compare their quantum statistical properties, including quadrature squeezing and antibunching effect.

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1. Introduction

Coherent states (CSs) of the harmonic oscillator [1] as well as generalized CSs associated with various algebras [2–4] play an important role in various fields of physics, and in particular, they have found considerable applications in quantum optics. The former, defined as the eigenstates of the annihilation operator \hat{a} , have properties similar to those of the classical radiation field. The latter, in contrast, may exhibit some nonclassical properties such as photon antibunching [5] or sub-Poissonian photon statistics [6] and squeezing [7], which have given rise to an ever-increasing interest during the last decade. Among the generalized CSs, the so-called nonlinear CSs or f -deformed CSs [8] have attracted much attention in recent years, mostly because they exhibit nonclassical properties, such as amplitude squeezing and quantum interference [9, 10]. These states, which could be realized in the centre-of-mass motion of an appropriately laser-driven trapped ion [10] and in a micromaser under intensity-dependent atom–field interaction [11], are associated with nonlinear algebras and defined as

the eigenstates of the annihilation operator of an f -deformed oscillator [8, 12]. Up to now, many quantum optical states such as q -deformed CSs [12], the binomial states (or displaced excited CSs) [13], negative binomial states [14, 15] and photon-added and photon-subtracted CSs [16, 17] have been considered and studied as some examples of nonlinear CSs.

One of the most important questions is the physical meaning of the deformation function in the nonlinear CSs. We have some theoretical and experimental data that consider the effect of curvature on the physical and optical properties of nano-structures ([18] and its references). On the other hand, it has been found [19] that there is a close connection between the deformation function appeared in the nonlinear CSs algebraic structure and the non-commutative geometry of the configuration space. Furthermore, it is interesting to point out that our recent studies [20] have revealed strictly physical relationship between the nonlinearity concept resulting from f -deformation and some nonlinear optical effects, e.g., Kerr nonlinearity, in the context of atom–field interaction. These witnesses led us to this idea that there is a relation between deformation function and curvature of the physical space. In other words, we are intended to investigate that is it possible to find different deformation functions in the nonlinear CSs corresponding to the geometry of space.

In the present contribution, our main purpose is to find a geometrical interpretation for the f -deformation function based on the curvature parameter of physical space. For this purpose, we investigate the influence of the curvature of the space on the algebraic structure of the CSs and their physical properties within the framework of nonlinear coherent states approach. We first study the algebras of the two-dimensional harmonic oscillator on the flat space and on a non-flat space (sphere) in section 2. In section 3, we show that these two oscillator algebras may be regarded as a one-dimensional f -deformed oscillator algebra. In particular, we find that the harmonic oscillator algebra on a sphere may be considered as a deformed version of oscillator algebra on the flat space. As an alternative point of view, in section 4 we show that the two-dimensional oscillator algebra on a sphere can be identified as a new type of deformed $su(2)$ algebra that in the limit of flat space it reduces to the standard $su(2)$ algebra. In section 5, we construct the corresponding CSs and examine their resolution of identity. Section 6 is devoted to the study of the quantum statistical properties of the constructed CSs. Especially, the influence of curvature of the physical space on their nonclassical properties is clarified. Finally, the summary and concluding remarks are given in section 7.

2. Deformed algebra for two-dimensional harmonic oscillator

2.1. Two-dimensional harmonic oscillator on flat space

The two-dimensional quantum harmonic oscillator in the flat space is described by the Hamiltonian

$$\hat{H} = \frac{1}{2}(\hat{p}_x^2 + \hat{p}_y^2) + \frac{1}{2}(\hat{x}^2 + \hat{y}^2) \quad (1)$$

(in this paper we put $\hbar = m = \omega = 1$). By using the Fradkin operators [21]

$$\hat{B} = \hat{S}_{xx} - \hat{S}_{yy} = (\hat{p}_x^2 + \hat{x}^2) - (\hat{p}_y^2 + \hat{y}^2), \quad \hat{S}_{xy} = \hat{p}_x \hat{p}_y + \hat{x} \hat{y} \quad (2)$$

and angular momentum operator

$$\hat{L} = \hat{x} \hat{p}_y - \hat{y} \hat{p}_x, \quad (3)$$

one can define the following operators:

$$\begin{aligned}
\hat{n} &= \frac{\hat{L}}{2} - u\hat{I}, \\
\hat{A}^\dagger &= \frac{1}{2} \left(\frac{\hat{B}}{2} + i\hat{S}_{xy} \right), \\
\hat{A} &= \frac{1}{2} \left(\frac{\hat{B}}{2} - i\hat{S}_{xy} \right),
\end{aligned} \tag{4}$$

where u is a constant to be determined. From (4) it follows that \hat{A} , \hat{A}^\dagger and \hat{n} satisfy the following closed algebraic relations [22]:

$$\begin{aligned}
[\hat{n}, \hat{A}^\dagger] &= \hat{A}^\dagger, \\
[\hat{n}, \hat{A}] &= -\hat{A}, \\
[\hat{A}, \hat{A}^\dagger] &= \Phi(\hat{H}, \hat{n} + 1) - \Phi(\hat{H}, \hat{n}).
\end{aligned} \tag{5}$$

The structure function, $\Phi(E, n) = \frac{1}{4}[E^2 - (2n + 2u - 1)^2]$, is a real definite positive function for $n \geq 0$ and $\Phi(E, 0) = 0$. With this algebra, we can define the corresponding Fock space for each energy eigenvalue,

$$\begin{aligned}
\hat{H}|E, n\rangle &= E|E, n\rangle, \\
\hat{n}|E, n\rangle &= n|E, n\rangle, \quad n = 0, 1, 2, \dots, \\
\hat{A}|E, 0\rangle &= 0, \\
|E, n\rangle &= \frac{1}{\sqrt{[\Phi(E, n)]!}} (\hat{A}^\dagger)^n |E, 0\rangle,
\end{aligned} \tag{6}$$

where, by definition

$$[0]! = 1, \quad [\Phi(E, n)]! = \Phi(E, n)[\Phi(E, n-1)]!. \tag{7}$$

In the case of the discrete energy eigenvalues, for every eigenvalue E there is some degeneracy of dimension $N + 1$ which is a root of the structure function. Therefore, the dimensionality of the Fock space corresponding to that energy eigenvalue should be equal to $N + 1$,

$$\begin{aligned}
\hat{H}|N, n\rangle &= E_N|N, n\rangle, \quad N = 0, 1, 2, \dots \\
\hat{n}|N, n\rangle &= n|N, n\rangle, \quad n = 0, 1, 2, \dots, N.
\end{aligned} \tag{8}$$

This restriction, combined with the annihilation of the structure function for $n = 0$, determines the energy eigenvalues

$$(E_N)_{\text{flat}} \equiv (E_N)_f = N + 1, \quad u = -\frac{N}{2}, \tag{9}$$

and therefore we find

$$\Phi_f(E_N, n) = n(N + 1 - n). \tag{10}$$

2.2. Two-dimensional harmonic oscillator on a sphere

There are several coordinate systems on a sphere which are useful generalizations of the Cartesian systems of Euclidean geometry. If we use the gnomonic projection, which is the projection onto the tangent plane from the centre of the sphere in the embedding space, and denote the Cartesian coordinates of this projection by x_i , the Hamiltonian of the harmonic oscillator on a sphere with curvature $\lambda = \frac{1}{R^2}$ may be written in the form [23]

$$\hat{H} = \frac{1}{2}(\hat{\pi}^2 + \lambda\hat{L}^2) + \frac{1}{2}(\hat{x}^2 + \hat{y}^2), \tag{11}$$

where

$$\vec{\pi} = \vec{p} + \frac{\lambda}{2} [\vec{x}(\vec{x} \cdot \vec{p}) + (\vec{p} \cdot \vec{x})\vec{x}], \quad (12)$$

and

$$\hat{L}^2 = \frac{1}{2} \hat{L}_{ij} \hat{L}_{ij}, \quad \hat{L}_{ij} = \hat{x}_i \hat{p}_j - \hat{x}_j \hat{p}_i. \quad (13)$$

Using the same method as for the flat harmonic oscillator, we obtain

$$(E_N)_{\text{sphere}} \equiv (E_N)_s = \sqrt{1 + \frac{\lambda^2}{4}}(N+1) + \frac{\lambda}{2}(N+1)^2, \quad u = \frac{N}{2}, \quad (14)$$

and the structure function can be written as

$$\Phi_s(E, n) = n(N+1-n)(\lambda(N+1-n) + \sqrt{1 + \lambda^2/4})(\lambda n + \sqrt{1 + \lambda^2/4}). \quad (15)$$

3. Two-dimensional harmonic oscillator as a deformed harmonic oscillator

The annihilation and creation operators associated with an f -deformed harmonic oscillator algebra satisfy the following commutation relation [12]:

$$[\hat{A}, \hat{A}^\dagger] = (\hat{n} + 1)f^2(\hat{n} + 1) - \hat{n}f^2(\hat{n}), \quad (16)$$

where

$$\hat{A} = \hat{a}f(\hat{n}) = f(\hat{n} + 1)\hat{a}, \quad \hat{A}^\dagger = f^\dagger(\hat{n})\hat{a}^\dagger = \hat{a}^\dagger f^\dagger(\hat{n} + 1). \quad (17)$$

\hat{a} , \hat{a}^\dagger and $\hat{n} = \hat{a}^\dagger \hat{a}$ are the bosonic annihilation, creation and number operators, respectively, and $f(\hat{n})$ is the deformation function. Ordinarily, the phase of f is irrelevant and we can choose f to be real and nonnegative, i.e. $f^\dagger(\hat{n}) = f(\hat{n})$.

On the other hand, according to (5), we can write

$$[\hat{A}, \hat{A}^\dagger] = \Phi(\hat{H}, \hat{n} + 1) - \Phi(\hat{H}, \hat{n}). \quad (18)$$

Thus, if we work with a constant energy, $E = E_N$, then $\Phi(\hat{H}, \hat{n})$ depends only on \hat{n} , and we can write

$$nf^2(n) = \Phi(E_N, n). \quad (19)$$

In the flat space where $\Phi_f(E_N, n)$ is given by (10), we have

$$f_f(\hat{n}) = \sqrt{(N+1-\hat{n})}, \quad (20)$$

while on the sphere where $\Phi_s(E_N, n)$ is given by (15) we have

$$f_s(n) = f_f(\hat{n})g(\lambda, n), \quad (21)$$

where

$$g(\lambda, n) = \sqrt{(\lambda(N+1-n) + \sqrt{1 + \lambda^2/4})(\lambda n + \sqrt{1 + \lambda^2/4})}. \quad (22)$$

It is obvious that in the flat limit, $g(\lambda, n) \rightarrow 1$ and (21) reduces to (20). Therefore, we can consider the two-dimensional harmonic oscillator algebra as a type of one-dimensional deformed harmonic oscillator with the deformation function $f(n)$ and also the sphere oscillator algebra as a deformed version of flat oscillator algebra with the deformation function $g(\lambda, n)$.

4. Two-dimensional harmonic oscillator algebra as a deformed $\mathfrak{su}(2)$ algebra

In this section, we show that the two-dimensional harmonic oscillator algebra can also be considered as a deformed $\mathfrak{su}(2)$ algebra. For this purpose, we begin with the commutation relation

$$[\hat{A}^\dagger, \hat{A}] = \hat{n} f^2(\hat{n}) - (\hat{n} + 1) f^2(\hat{n} + 1), \quad (23)$$

(cf equation (16)). According to (20), the above relation in the flat space takes the following form:

$$[\hat{A}^\dagger, \hat{A}] = 2 \left(\hat{n} - \frac{N}{2} \right). \quad (24)$$

If we now make the identifications

$$\begin{aligned} (\hat{A}^\dagger) &\rightarrow \hat{J}_+ \\ (\hat{A}) &\rightarrow \hat{J}_- \\ \left(\hat{n} - \frac{N}{2} \right) &\rightarrow \hat{J}_0, \end{aligned} \quad (25)$$

we arrive at the standard $\mathfrak{su}(2)$ algebra

$$[\hat{J}_0, \hat{J}_\pm] = \hat{J}_\pm, \quad [\hat{J}_+, \hat{J}_-] = 2\hat{J}_0, \quad (26)$$

for which the eigenstates are $|j, m\rangle$ s such that

$$\hat{J}_0 |j, m\rangle = m |j, m\rangle, \quad -j \leq m \leq j. \quad (27)$$

If we now compare this relation with (8), we find that ns are positive while ms may take both negative and positive values. According to positivity of the values of n , which denote the number of excitation quanta, the identification of the above two algebras requires that a constant value $\left(\frac{2j}{2} = \frac{N}{2}\right)$ is added to the eigenvalues m , so that they become positive. This is a reason for the identification $\left(\hat{n} - \frac{N}{2}\right) \rightarrow \hat{J}_0$.

On the other hand, if we use the deformation function for the two-dimensional harmonic oscillator algebra on a sphere (equation (21)) and if we use the same identification as (25), we have

$$[\hat{J}_0, \hat{J}_\pm] = \hat{J}_\pm, \quad [\hat{J}_+, \hat{J}_-] = 2\hat{J}_0 h(\lambda, N, \hat{J}_0), \quad (28)$$

where

$$h(\lambda, N, \hat{J}_0) = \left\{ 1 + \lambda \left(1 + \frac{\lambda}{4} \right)^{1/2} (N + 1) - \lambda^2 \left[2\hat{J}_0^2 - N \left(\frac{N}{2} + 1 \right) - \frac{1}{4} \right] \right\}. \quad (29)$$

It is clear that in the flat limit, $\lambda \rightarrow 0$, $h(\lambda, N, \hat{J}_0) \rightarrow 1$, and the above deformed $\mathfrak{su}(2)$ algebra reduces to the standard non-deformed $\mathfrak{su}(2)$ algebra. Thus, we conclude that two oscillator algebras (two W-H algebras) in the flat space with this approach can be regarded as an $\mathfrak{su}(2)$ algebra (like the Schwinger model) and also these two algebras can be considered as a deformed $\mathfrak{su}(2)$ algebra on a sphere. This deformed $\mathfrak{su}(2)$ algebra is one of the generalized deformed $\mathfrak{su}(2)$ algebras defined in [28].

5. Coherent states in the finite-dimensional Hilbert space

By using the definition of deformed creation and annihilation operators (equations (17)) and the deformation functions $f_f(\hat{n})$ and $f_s(\hat{n})$, we find

$$\hat{A}|0\rangle = 0 = \hat{A}^\dagger|N\rangle. \quad (30)$$

Thus, for each constant value of N (or constant value of energy E_N), we encounter with a finite-dimensional Hilbert space. In this section, we are intended to construct coherent states associated with the flat space and sphere.

5.1. CSs on the flat space

On the flat space, we have

$$\hat{A}|n\rangle = \chi_n^N |n-1\rangle, \quad \hat{A}^\dagger |n\rangle = \chi_{n+1}^N |n+1\rangle, \quad (31)$$

where

$$\chi_n^N = \sqrt{n(N+1-n)}. \quad (32)$$

We can make use of the formalism of constructing truncated coherent states [27] and define the finite-dimensional CSs as

$$|\mu\rangle_f = (1 + |\mu|^2)^{-N/2} \exp(\mu \hat{A}^\dagger) |0\rangle = (1 + |\mu|^2)^{-N/2} \sum_{n=0}^N \sqrt{\binom{N}{n}} \mu^n |n\rangle, \quad (33)$$

where μ is a complex number. It is seen that these CSs are analogous to the spin CSs that are constructed by using the $\mathfrak{su}(2)$ algebra.

5.2. CSs on a sphere

From (17) and (21), we have

$$\hat{A}|n\rangle = [g(\lambda, n)] \chi_n^N |n-1\rangle, \quad \hat{A}^\dagger |n\rangle = [g(\lambda, n+1)] \chi_{n+1}^N |n+1\rangle. \quad (34)$$

Therefore, as in the case of the flat space we can define the associated CSs on sphere,

$$|\mu\rangle_s = C \exp(\mu \hat{A}^\dagger) |0\rangle = C \sum_{n=0}^N \sqrt{\binom{N}{n}} [g(\lambda, n)]! \mu^n |n\rangle, \quad (35)$$

where

$$\frac{1}{C^2} = \sum_{n=0}^N \binom{N}{n} \{[g(\lambda, n)]!\}^2 (|\mu|^2)^n. \quad (36)$$

It is found that for small values of λ , we have

$$g(\lambda, n) = 1 + \frac{\lambda}{2}(N+1) + o(\lambda^2), \quad (37)$$

so that

$$[g(\lambda, n)]! = \left[1 + \frac{\lambda}{2}(N+1)\right]^n. \quad (38)$$

Thus, in this limit we obtain

$$|\mu\rangle_s = C \sum_{n=0}^N \sqrt{\binom{N}{n}} \left[\mu \left(1 + \frac{\lambda}{2}(N+1)\right)\right]^n |n\rangle. \quad (39)$$

In other words, for obtaining the CSs, it is sufficient to make the replacement $\mu \rightarrow \mu(1 + \frac{\lambda}{2}(N+1))$ for the CSs in the flat space in the large radius limit.

5.3. Resolution of identity

In this section, we show that the CSs on the flat space and on a sphere form an overcomplete set. Since it is necessary to include a measure function $m(|\mu|^2)$ in the integral, we require

$$\int d^2\mu |\mu\rangle m(|\mu|^2) \langle\mu| = \sum_{n=0}^N |n\rangle \langle n| = \mathbb{I}. \tag{40}$$

In the case of flat space,

$$\begin{aligned} \int d^2\mu |\mu\rangle_f m_f(|\mu|^2)_f \langle\mu| &= \sum_{n=0}^N |n\rangle \langle n| \binom{N}{n} \int \frac{1}{2} d(|\mu|^2) d\theta \frac{|\mu|^{2n}}{(1+|\mu|^2)^N} m_f(|\mu|^2) \\ &= \pi \sum_{n=0}^N |n\rangle \langle n| \binom{N}{n} \int_0^\infty d(|\mu|^2) \frac{|\mu|^{2n}}{(1+|\mu|^2)^N} m_f(|\mu|^2). \end{aligned}$$

Thus, we should have

$$\int_0^\infty d(|\mu|^2) \frac{|\mu|^{2n}}{(1+|\mu|^2)^N} m_f(|\mu|^2) = \frac{1}{\pi \binom{N}{n}}. \tag{41}$$

The suitable choice for the measure function reads [26]

$$m_f(|\mu|^2) = \frac{N+1}{\pi} \frac{1}{(1+|\mu|^2)^2}. \tag{42}$$

In this manner, the resolution of identity is

$$\frac{N+1}{\pi} \int \frac{d^2\mu}{(1+|\mu|^2)^2} |\mu\rangle_f \langle\mu| = \mathbb{I}. \tag{43}$$

In order to examine the resolution of identity for the CSs on a sphere, we first introduce deformed Binomial expansion

$$(1+x)_\lambda^N = \sum_{n=0}^N \binom{N}{n}_\lambda x^n, \tag{44}$$

where by definition

$$\binom{N}{n}_\lambda = \binom{N}{n} \{[g(\lambda, n)]!\}^2. \tag{45}$$

We see that when $\lambda \rightarrow 0$, $g(\lambda, n) \rightarrow 1$ and the deformed Binomial expansion becomes the well-known Binomial expansion. With the use of this definition, we can write (35) as

$$|\mu\rangle_s = (1+|\mu|^2)_\lambda^{-N/2} \sum_{n=0}^N \sqrt{\binom{N}{n}_\lambda} \mu^n |n\rangle. \tag{46}$$

For the resolution of identity, we should have

$$\int d^2\mu |\mu\rangle_s m_s(|\mu|^2)_s \langle\mu| = \sum_{n=0}^N |n\rangle \langle n| = \mathbb{I} \tag{47}$$

or

$$\pi \sum_{n=0}^N |n\rangle \langle n| \binom{N}{n}_\lambda \int_0^\infty d(|\mu|^2) \frac{|\mu|^{2n}}{(1+|\mu|^2)_\lambda^N} m_s(|\mu|^2) = \mathbb{I}. \tag{48}$$

If we define the corresponding measure as

$$m_s(|\mu|^2) = \frac{N+1}{\pi} \frac{1}{(1+|\mu|^2)_\lambda^2}, \quad (49)$$

and the deformed version of integral (41) as

$$\int_{0(\lambda)}^{\infty} d(|\mu|^2) \frac{|\mu|^{2n}}{(1+|\mu|^2)_\lambda^N} m_s(|\mu|^2) = \frac{1}{\pi \binom{N}{n}_\lambda}, \quad (50)$$

then we obtain the resolution of identity for the CSs on a sphere,

$$\frac{N+1}{\pi} \int_{(\lambda)} \frac{d^2\mu}{(1+|\mu|^2)_\lambda^2} |\mu\rangle_{s,s} \langle\mu| = \mathbb{I}. \quad (51)$$

6. Quantum statistical properties of the flat and sphere CSs

Theoretically, we describe an ideal laser by the standard (Glauber) CSs. In particular, the photon-number distribution of this ideal laser, like the non-deformed CSs, is exactly Poissonian [25]. But the photon-number statistics of real lasers do not coincide with this description. It has been shown [29] that, when the photon-number distribution is concerned, the q -deformed CSs are more suitable states for describing nonideal lasers and other nonlinear interactions such as photons emitted by single-atom resonance fluorescence that we can say these photon states as deformed (or dressed) photon states.

In the present section, we shall proceed to study some quantum statistical properties of the constructed CSs, including mean number of photons, Mandel parameter and quadrature squeezing.

6.1. Photon-number distribution

The probability of finding n quanta in the flat CSs, i.e., its photon-number distribution is given by

$$P_f(n, \mu, N) = (1+|\mu|^2)^{-N} \binom{N}{n} \mu^{(2n)}. \quad (52)$$

The mean number of photons in the state $|\mu\rangle_f$ is equal to

$$\langle \hat{n} \rangle_f = \langle \mu | \hat{a}^\dagger \hat{a} | \mu \rangle_f = \sum_{n=0}^N n P_f(n, \mu, N). \quad (53)$$

The photon-number distribution for the CSs on the sphere is given by

$$P_s(n, \mu, N, \lambda) = \frac{1}{\sum_{n=0}^N \binom{N}{n} [g(\lambda, n)!]^2 \mu^{(2n)}} \binom{N}{n} [g(\lambda, n)!]^2 \mu^{(2n)}, \quad (54)$$

and the mean number of photons in the state $|\mu\rangle_s$ is equal to

$$\langle \hat{n} \rangle_s = \langle \mu | \hat{a}^\dagger \hat{a} | \mu \rangle_s = \sum_{n=0}^N n P_s(n, \mu, N). \quad (55)$$

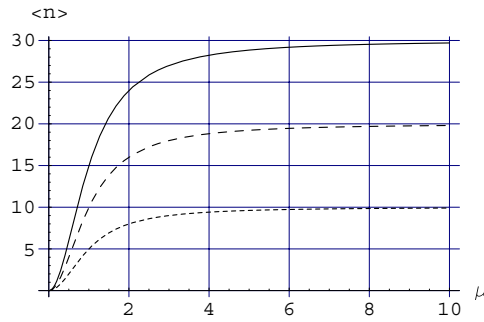


Figure 1. Mean number of photons in the state $|\mu\rangle_f$ versus μ , the dotted corresponds to $N = 10$, the dashed to $N = 20$ and the solid curve to $N = 30$.

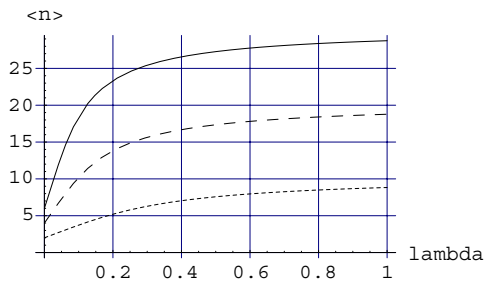


Figure 2. Mean number of photons in the state $|\mu\rangle_s$ versus λ for $\mu = 0.5$, the dotted corresponds to $N = 10$, the dashed to $N = 20$ and the solid curve to $N = 30$.

In figure 1, we have plotted the variation of mean number of photons in the state $|\mu\rangle_f$ with respect to μ for different values of N . It is seen that for a given N , the mean number of photons is increased by increasing μ and in the limit of $\mu \rightarrow \infty$ we have

$$\lim_{\mu \rightarrow \infty} \langle \mu | \hat{n} | \mu \rangle_f \rightarrow N, \tag{56}$$

where N is the dimension of the Hilbert space.

Figure 2 displays the mean number of photons for the CSs on a sphere with respect to λ for $N = 10, 20, 30$ with $\mu = 0.5$. As it is seen, for a fixed value of N , the mean number of photons is increased by increasing λ . It is in agreement with relation (39) for small λ .

Since for the non-deformed coherent states the variance of the number operator is equal to its average, deviations from Poisson distribution can be measured with the Mandel parameter [24]

$$M = \frac{(\Delta n)^2 - \langle \hat{n} \rangle}{\langle \hat{n} \rangle}, \tag{57}$$

which is negative for a sub-Poissonian distribution (photon antibunching) and positive for a super-Poissonian distribution (photon bunching). In figure 3, we show the effect of curvature on the variation of Mandel parameter with $\mu = 0.5$ and for different values of N . We see that when we go from flat space to sphere, photon-counting statistics of the CSs tends to sub-Poissonian more rapidly. In other words, the CSs on the sphere show more nonclassical properties than the CSs on the flat space.

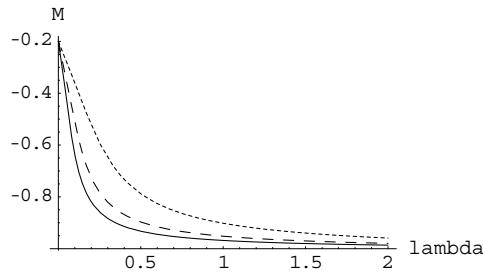


Figure 3. Mandel parameter versus λ for $\mu = 0.5$, the dotted corresponds to $N = 10$, the dashed to $N = 20$ and the solid curve to $N = 30$.

6.2. Quadrature squeezing

Let us consider the conventional quadrature operators X_{1a} and X_{2a} defined in terms of non-deformed operators \hat{a} and \hat{a}^\dagger [25],

$$\hat{X}_{1a} = \frac{1}{2}(\hat{a} e^{i\varphi} + \hat{a}^\dagger e^{-i\varphi}), \quad \hat{X}_{2a} = \frac{1}{2i}(\hat{a} e^{i\varphi} - \hat{a}^\dagger e^{-i\varphi}). \quad (58)$$

The commutation relation for \hat{a} and \hat{a}^\dagger leads to the following uncertainty relation:

$$(\Delta X_{1a})^2 (\Delta X_{2a})^2 \geq \frac{1}{16} |\langle [\hat{X}_{1a}, \hat{X}_{2a}] \rangle|^2 = \frac{1}{16}. \quad (59)$$

In the vacuum state $|0\rangle$, we have $(\Delta X_{1a})_0^2 = (\Delta X_{2a})_0^2 = \frac{1}{4}$ and so $(\Delta X_{1a})_0^2 (\Delta X_{2a})_0^2 = \frac{1}{16}$. While it is impossible to lower the product $(\Delta X_{1a})^2 (\Delta X_{2a})^2$ below the vacuum uncertainty value, it is nevertheless possible to define squeezed states for which at most one quadrature variance lies below the vacuum value,

$$(\Delta X_{ia})^2 < (\Delta X_{ia})_0^2 = \frac{1}{4} \quad (i = 1 \text{ or } 2), \quad (60)$$

or

$$S_{ia}(\varphi) = 4(\Delta X_{ia})^2 - 1 < 0. \quad (61)$$

Let us also consider here the deformed quadrature operators \hat{X}_{1A} and \hat{X}_{2A} defined in terms of deformed operators \hat{A} and \hat{A}^\dagger ,

$$\hat{X}_{1A} = \frac{1}{2}(\hat{A} e^{i\varphi} + \hat{A}^\dagger e^{-i\varphi}), \quad \hat{X}_{2A} = \frac{1}{2i}(\hat{A} e^{i\varphi} - \hat{A}^\dagger e^{-i\varphi}). \quad (62)$$

The commutation relation for \hat{A} and \hat{A}^\dagger (equation (16)) leads to the following uncertainty relation:

$$(\Delta X_{1A})^2 (\Delta X_{2A})^2 \geq \frac{1}{16} |[\langle \hat{X}_{1A}, \hat{X}_{2A} \rangle]|^2 = \frac{1}{16} (\langle (\hat{n} + 1) f^2(\hat{n} + 1) - \hat{n} f^2(\hat{n}) \rangle)^2. \quad (63)$$

In this case, the condition of quadrature squeezing reads

$$(\Delta X_{iA})^2 < \frac{1}{4} \langle (\hat{n} + 1) f^2(\hat{n} + 1) - \hat{n} f^2(\hat{n}) \rangle \quad (i = 1 \text{ or } 2) \quad (64)$$

or, equivalently,

$$S_{iA} = 4(\Delta X_{iA})^2 - \langle (\hat{n} + 1) f^2(\hat{n} + 1) \rangle + \langle \hat{n} f^2(\hat{n}) \rangle < 0. \quad (65)$$

In figures 4(a) and (b), respectively, we have plotted S_{1a} and S_{2a} with respect to φ for $N = 10$ and different values of λ . These figures clearly show that by increasing λ the quadrature squeezing is enhanced.

Figures 5(a) and (b), respectively, show S_{1A} and S_{2A} for $N = 10$ and different values of λ . Here, we again find that by increasing λ the degree of squeezing is enhanced. However,

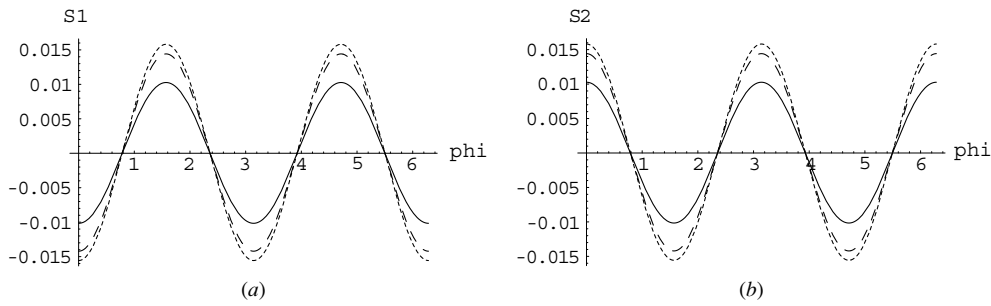


Figure 4. (a) S_{1a} versus φ for $N = 10$ and $\mu = 0.1$, the solid corresponds to $\lambda = 0.0$, the dashed to $\lambda = 0.05$ and the dotted curve to $\lambda = 0.1$. (b) S_{2a} versus φ for $N = 10$ and $\mu = 0.1$, the solid corresponds to $\lambda = 0.0$, the dashed to $\lambda = 0.05$ and the dotted curve to $\lambda = 0.1$.

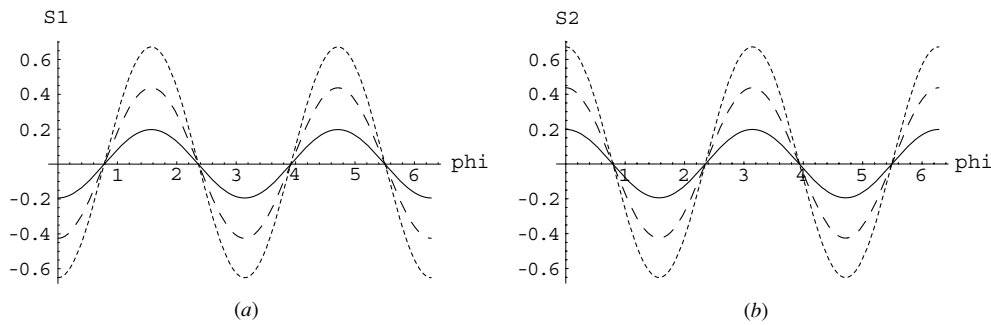


Figure 5. (a) S_{1A} versus φ for $N = 10$ and $\mu = 0.1$, the solid corresponds to $\lambda = 0.0$, the dashed to $\lambda = 0.05$ and the dotted curve to $\lambda = 0.1$. (b) S_{2A} versus φ for $N = 10$ and $\mu = 0.1$, the solid corresponds to $\lambda = 0.0$, the dashed to $\lambda = 0.05$ and the dotted curve to $\lambda = 0.1$.

in comparison with the non-deformed quadratures, deformed quadratures exhibit stronger squeezing and increasing of λ leads to much pronounced squeezing. In summary, we can result that by going from the flat space to sphere the CSs exhibit increased nonclassical properties.

7. Summary and concluding remarks

In this paper, we have searched for a relation between the deformation function of the f-deformed oscillator algebra and two-dimensional harmonic oscillator on the flat space and on sphere. We have found that we could consider two-dimensional harmonic oscillator algebra as a deformed one-dimensional harmonic oscillator algebra. We have obtained two deformation function corresponding to the flat and sphere harmonic oscillators and we have also shown that sphere deformation function is a g-deformed with respect to the flat deformation function. Furthermore, we have found that the two-dimensional harmonic oscillator algebra on a sphere may be considered as a deformed $su(2)$ with a deformation function corresponding to the curvature of space that in the flat limit tends to unity. Then, we have constructed the CSs for these spaces and studied their quantum statistical properties. The results show that the curvature of physical space leads to the enhancement of nonclassical properties of the states.

We are now searching for a physical model that generates our coherent states on the flat and spherical surfaces. At the present step what we can conjecture is that photons on

the sphere may be interpreted physically as dressed photons [29] that are equivalent to bare photons plus a physical object (like a physical field). If we find that model, we can provide reasonable answer to the relevant questions about the physical meaning of the photons on the curved space. The work is in progress.

Furthermore, the present contribution may be considered as the first step for finding the relation between nonlinearity of CSs and curvature of the physical space (here we have worked only with sphere). As another direction of the work, we have started on generalizing our approach to other non-flat surfaces, e.g. the surface with negative curvature, in order to find a complete relation between geometry and deformation function together with its physical implications.

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